*Int. J. Heat Mass Transfer.* Vol. 32, No. 5, pp. 977–980, 1989
Printed in Great Britain

0017-9310/89 $3.00+0.00


# Two-dimensional fin with non-constant root temperature

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(*Received* 6 *April* 1988 *and in final form* 11 *October* 1988)

## INTRODUCTION

A COMMON simplification made when analyzing an extended surface is that the temperature within the fin is a function of one dimension (measured directly away from the root ($x'$)). Many papers [1–6] have shown this one-dimensional approach, albeit convenient, may be in error under certain physical conditions (e.g. when the convection coefficient, $h$, is large compared to the fin material thermal conductivity). Typically the validity criterion is described by the root Biot number magnitude—for constant $h$, the criterion for the validity of the one-dimensional assumption is that the Biot number, based on the half thickness of the fin, be much less than one.

Another situation where the usual one-dimensional assumption is particularly in error is when the convection coefficients of the heat transfer surfaces (other than the tip) are not equal. An example of this would be a horizontal fin in a natural convection environment. That is, even though many examples have been presented in which the magnitude of the convection coefficients at the fin tip and the other heat transfer surfaces are different, the one-dimensional restriction does not allow the convection coefficient to be different for those other surfaces. Further, listed in any heat transfer book are experimental correlations for hot surfaces facing up and hot surfaces facing down [7]. The resulting natural convection coefficient magnitudes may differ by a factor of 2 when all other parameters are equal. Thus even though the convection coefficient, $h$, is not truly constant from the root to the tip in a real fin [8, 9], $h$ (top surface) is greater than or, at the best, equal to $h$ (bottom surface).

Finally, the actual base temperature of the fin is not really constant [10, 11] as is usually assumed in analytical work [12]. This fin root temperature elevation or depression results in errors in the calculated magnitudes of heat lost from the fin.

This note presents the results of an investigation of the effects of an idealized non-constant fin root temperature on the heat lost from the fin when the convection coefficients of all surfaces are not equal and thermal radiation effects are neglected. Thus the purpose of this note is to provide insight into the effects of non-constant root temperature and unequal top, bottom, and tip surface convection coefficients. This difference will be denoted by the Biot numbers, $B$, rather than the convection coefficients. The restrictions of this analysis are (1) $0 \leqslant B_2 \leqslant B_1 \leqslant 1$, (2) $B_1 = 1$, 0.1 or 0.01, (3) $0 \leqslant B_3 \leqslant 100$, and (4) the non-constant root temperature variation is at most $\pm 20\%$.

## TWO-DIMENSIONAL ANALYSIS

In the case of a two-dimensional rectangular fin and constant physical properties, the governing equation and boundary conditions are

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \tag{1}$$

$$x=0,\quad \theta=\theta_0+a\cos\left(\frac{\pi y}{2}\right) \tag{2}$$

$$x=L,\quad \frac{\partial\theta}{\partial x}+B_3\theta=0 \tag{3}$$

(2) and (3): $-1 \leqslant y \leqslant 1$

$$y=1,\quad \frac{\partial\theta}{\partial y}+B_1\theta=0 \tag{4}$$

$$y=-1,\quad \frac{\partial\theta}{\partial y}-B_2\theta=0 \tag{5}$$

(4) and (5): $0 \leqslant x \leqslant L$.

Figure 1 illustrates the geometry of this problem. The solution of equation (1) using the boundary conditions of equations (2)–(5) is

$$\theta=\sum_{n=1}^{\infty} f_1(y)f_2(x)N_n \tag{6}$$

and the heat lost per fin length in this two-dimensional case is

$$Q=\int_{-l}^{l}\left[-k\frac{\partial T}{\partial x}\right]_{x=0}\mathrm{d}y' = 2k\sum_{n=1}^{\infty}\sin(\lambda_n)f_nN_n \tag{7}$$

where

$$f_1(y)=\cos(\lambda_n y)+(A_n)\sin(\lambda_n y) \tag{8}$$

$$f_2(x)=\cosh(\lambda_n x)+(f_n)\sinh(\lambda_n x) \tag{9}$$

$$N_n=[2\theta_0\sin(\lambda_n)/(D_n\lambda_n)][1+2b\pi\lambda_n\cot(\lambda_n)/(\pi^2-4\lambda_n^2)] \tag{10}$$

$$D_n=1+\sin(2\lambda_n)/(2\lambda_n)+A_n^2[1-\sin(2\lambda_n)/(2\lambda_n)] \tag{11}$$

$$f_n=[B_3+\lambda_n\tanh(\lambda_nL)]/[\lambda_n+B_3\tanh(\lambda_nL)] \tag{12}$$

$$A_n=\frac{(\lambda_n)\tan(\lambda_n)-B_1}{(\lambda_n)+B_1\tan(\lambda_n)}=\frac{-(\lambda_n)\tan(\lambda_n)+B_2}{(\lambda_n)+B_2\tan(\lambda_n)} \tag{13}$$

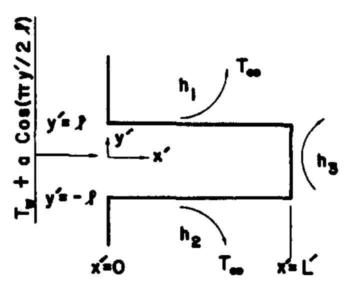


FIG. 1. Geometry of a thermally asymmetric, constant cross-sectional area, rectangular fin.

## NOMENCLATURE

| | |
|---|---|
| $a$ | maximum root depression (elevation) temperature |
| $b$ | $a/\theta_0$ |
| $B$ | Biot number, $hl/k$ |
| $h$ | convection coefficient |
| $k$ | thermal conductivity |
| $l$ | one half fin thickness |
| $L'$ | fin length |
| $L$ | $L'/l$ |
| $Q$ | power lost by the fin per length along the root under steady-state conditions [W m$^{-1}$] |
| $T$ | fin temperature [°C] |
| $x'$ | along the fin variable (root to tip) $\leqslant L$ |
| $x$ | $x'/l$ |
| $y'$ | across the fin variable $\leqslant \lvert \pm l \rvert$ |
| $y$ | $y'/l$ |
| $z'$ | fin depth. |

Greek symbols

| | |
|---|---|
| $\varepsilon$ | thermal asymmetry factor, $(B_1 - B_2)/B_1$ |
| $\theta$ | adjusted fin temperature excess, $T - T_\infty$ |
| $\theta_0$ | $T_w - T_\infty$ |
| $\lambda'$ | eigenfunction |
| $\lambda$ | $\lambda' l$. |

Subscripts

| | |
|---|---|
| 1 | one-dimensional or, if two-dimensional, top fin surface |
| 2 | two-dimensional or bottom fin surface |
| 3 | fin tip surface |
| w | wall or root |
| $\infty$ | ambient. |

and

$$b = a/\theta_0.$$

The values of $\lambda_n$ come from the roots of the last two expressions of equation (13).

## RESULTS

A Newton–Raphson method was used to determine the eigenfunctions of equation (13). The first six values of the results obtained when $B_1 = B_2$ were in exact agreement with tabulated values in ref. [7]. Further, equations (6) and (7) were used to compute the temperature profiles and heat transfer per fin length. In each case, the indicated summation process was terminated when the last term computed contributed less than 0.00001.

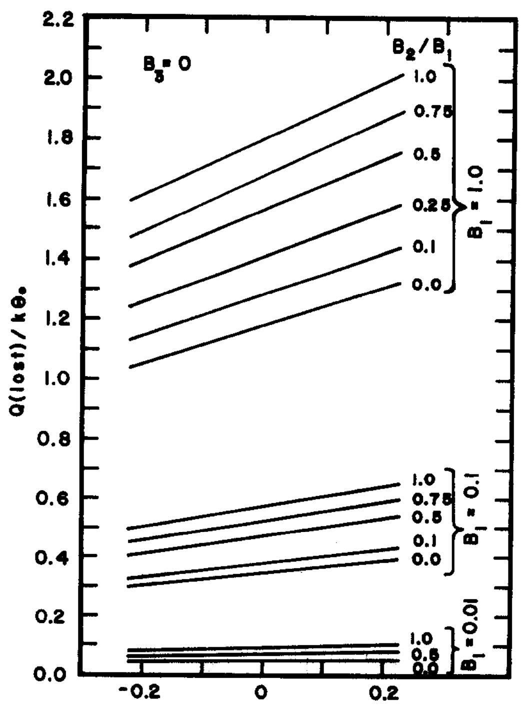


FIG. 2. Non-dimensional heat lost from a fin vs root temperature deviation in the case of $L = 5$ ($B_3 = 0$).

Figure 2 illustrates the non-constant root temperature effects on the heat lost from a fin for various asymmetry conditions. The parameters in this figure are the Biot numbers of the fin top surface, $B_1$, the fin bottom surface, $B_2$, and the tip, $B_3$. The situations selected for comparison are (1) an insulated tip denoted by $B_3 = 0$, (2) $B_3 = 0.25$, a convenient but arbitrarily selected value, and (3) the tip temperature equal to the ambient temperature. This last situation is represented as $B_3 \to \infty$ (i.e. $B_3(T(x = L) - T_\infty)$ must be finite). The actual calculations made were for a maximum value of $B_3 = 100$. In all cases presented, the tip effects are not negligible at small $B_1$ values because the corresponding curves for $B_3 = 0.25$ and 100 are the same shape but shifted up considerably (Table 1 may be used to compare the $b = 0$ values for the three $B_3$ values).

Some interesting information is obtained if a comparison is made of the heat lost by the fin when $B_1 = B_2$ and the heat lost by the fin when $B_1 > B_2$ (i.e. $Q^-$ to $Q^{\neq}$, respectively). Recalling that $Q^-/Q^{\neq} \geqslant 1$ for the parameter range of this study, Fig. 3 presents a comparison of this ratio for $b = 0$. Though not presented, the $b = \pm 0.2$ plots were virtually identical to those of Fig. 3. Thus even though $Q(\text{lost})/k\theta_0$ may vary considerably with $b$, the ratio $Q^-/Q^{\neq}$ is very nearly constant as $b$ is varied. Note that for the $B_3 = 0$ case, as $B_1$ increases, the magnitude of the ratio $Q^-/Q^{\neq}$ decreases for a given $B_2$. On the other hand, when $B_3 = 0.25$ (and larger values), the ratio increases as $B_1$ increases for a given $B_2$ (Table 1 lists the values).

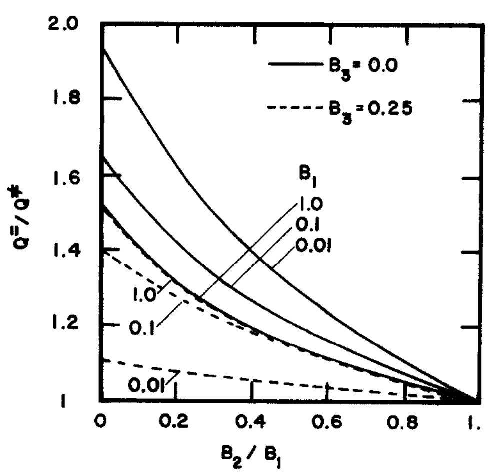


FIG. 3. Heat loss ratio in the case of $B_1 = B_2$ and $B_1 \geqslant B_2$ vs $B_2/B_1$ for $L = 5$, $b = 0$ and $B_3 = 0$ and 0.25.

Table 1. Comparison of $Q(\text{lost})/k\theta_0$, the slope of $Q(\text{lost})/k\theta_0$ vs $b$, and the values of $Q^-/Q^*$ for $b = 0$

| | | $B_1 = 1$ | | | $B_1 = 0.1$ | | | $B_1 = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|
| $B_1/B_2$ | $\frac{Q(\text{lost})}{k\Delta T}$ | Slope | $Q^-/Q^*$ | $\frac{Q(\text{lost})}{k\Delta T}$ | Slope | $Q^-/Q^*$ | $\frac{Q(\text{lost})}{k\Delta T}$ | Slope | $Q^-/Q^*$ |
| $B_3 = 0$ | | | | | | | | | |
| 1.0 | 1.8035 | 0.9383 | 1.0000 | 0.5703 | 0.3363 | 1.0000 | 0.09215 | 0.0560 | 1.0000 |
| 0.75 | 1.6824 | 1.0807 | 1.0720 | 0.5227 | 0.3348 | 1.0911 | 0.0814 | 0.0495 | 1.1320 |
| 0.5 | 1.5693 | 0.8641 | 1.1493 | 0.4711 | 0.3016 | 1.2106 | 0.0704 | 0.0428 | 1.3085 |
| 0.25 | 1.4115 | 0.7840 | 1.2777 | — | — | — | 0.0592 | 0.0360 | 1.5567 |
| 0.1 | 1.2865 | 0.7195 | 1.4018 | 0.3712 | 0.2391 | 1.5242 | 0.0523 | 0.03185 | 1.7607 |
| 0 | 1.1827 | 0.6643 | 1.5249 | 0.3461 | 0.2212 | 1.6478 | 0.04771 | 0.2904 | 1.9314 |
| $B_3 = 0.25$ | | | | | | | | | |
| 1.0 | 1.8038 | 0.9385 | 1.0000 | 0.6174 | 0.3665 | 1.0000 | 0.2747 | 0.1723 | 1.0000 |
| 0.75 | 1.6828 | 1.0810 | 1.0719 | 0.5779 | 0.3702 | 1.0684 | 0.2684 | 0.1686 | 1.0236 |
| 0.50 | 1.5700 | 0.8646 | 1.1489 | 0.5366 | 0.3435 | 1.1506 | 0.2620 | 0.1648 | 1.0487 |
| 0.25 | 1.4113 | 0.7852 | 1.2763 | — | — | — | 0.2555 | 0.1610 | 1.0754 |
| 0.10 | 1.2900 | 0.7217 | 1.3983 | 0.4628 | 0.2958 | 1.3339 | 0.2515 | 0.1587 | 1.0916 |
| 0 | 1.1887 | 0.6682 | 1.5174 | 0.4424 | 0.2828 | 1.3955 | 0.2489 | 0.1571 | 1.1038 |
| $B_3 = 100$ | | | | | | | | | |
| 1.0 | 1.8048 | 0.9391 | 1.0000 | 0.6810 | 0.4073 | 1.0000 | 0.4320 | 0.2725 | 1.0000 |
| 0.75 | 1.6842 | 1.0820 | 1.0716 | 0.7323 | 0.4158 | 1.0412 | 0.4279 | 0.2702 | 1.0094 |
| 0.50 | 1.5724 | 0.8662 | 1.1478 | 0.6169 | 0.3949 | 1.1039 | 0.4239 | 0.2680 | 1.0191 |
| 0.25 | 1.4180 | 0.7882 | 1.2728 | — | — | — | 0.4198 | 0.2657 | 1.0290 |
| 0.10 | 1.2982 | 0.7271 | 1.3902 | 0.5617 | 0.3590 | 1.2125 | 0.4173 | 0.2643 | 1.0351 |
| 0 | 1.2017 | 0.6764 | 1.5019 | 0.5469 | 0.3495 | 1.2454 | 0.4157 | 0.2634 | 1.0392 |

Values for $b = \pm 0.2$ are very nearly equal to the entries in this table for $Q^-/Q^*$. $Q(\text{lost})/k\theta_0$ varies as indicated in Fig. 2 and the slopes are exactly the same.

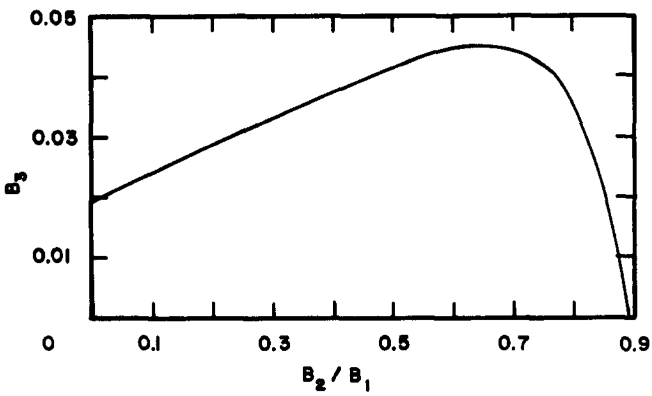


FIG. 4. $B_3$ vs $B_2/B_1$ for $L = 5$ when $(d/dB_1)(Q^-/Q^*) = 0$.

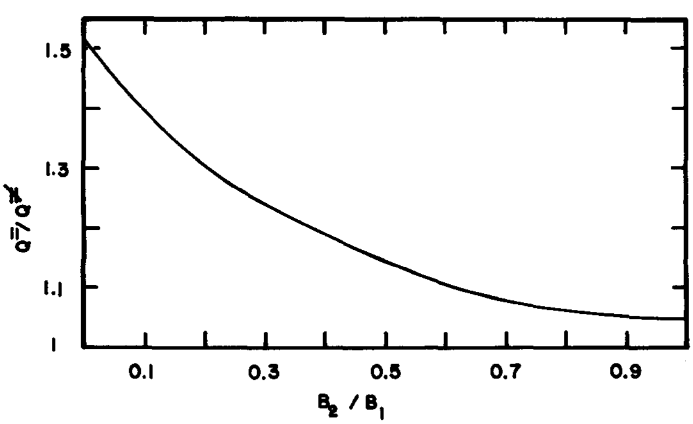


FIG. 5. Heat loss ratio in the case of $B_1 = B_2$ and $B_1 \geqslant B_2$ vs $B_2/B_1$ for $L = 5$ when $(d/dB_1)(Q^-/Q^*) = 0$.

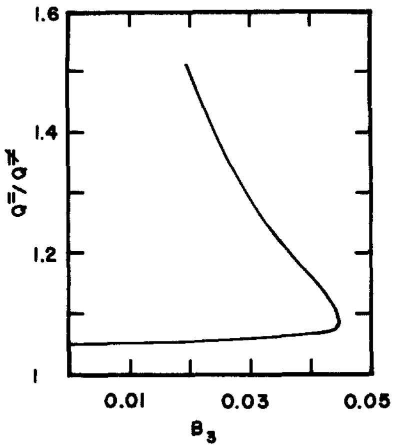


FIG. 6. Heat loss ratio in the case of $B_1 = B_2$ and $B_1 \geqslant B_2$ vs $B_3$ for $L = 5$ when $(d/dB_1)(Q^=/Q^{\neq}) = 0$.

The interesting question appears to be, "What value of $B_3$ produces a constant $Q^=/Q^{\neq}$ regardless of $B_1$ and what is the corresponding variation of $Q^=/Q^{\neq}$ with $B_2/B_1$?" Figure 4 presents $B_3$ as a function of $B_2/B_1$ such that the derivative of $Q_2^=/Q_2^{\neq}$ with respect to $B_1$ is zero (the zero slope restriction). Figure 5 illustrates $Q_2^=/Q_2^{\neq}$ vs $B_2/B_1$ for the same restriction. Finally Fig. 6 presents the variation of $Q^=/Q^{\neq}$ vs $B_3$ for the zero slope restriction. These graphs were produced from a numerical study, rather than a more direct extremizing procedure, because of mathematical difficulty.

## CONCLUSIONS

Perusal of the figures indicates that the fin tip condition does not have a strong effect on the thermal characteristics of a fin; the unequal surface Biot numbers do. Also the magnitude of the root temperature variation, $b$, is a very important parameter to the heat lost from a fin for large values of $B_1$. That is, for $B_1 = 1$, $Q(\text{lost})/k\theta_0$ (and $b = 0.2$) is directly proportional to $Q(\text{lost})/k\theta_0$ (and $b = -0.2$) with the proportionality constant of 1.23 at $B_2 = 1$. The magnitude of this constant increases slightly to 1.25 at $B_2 = 0$. Further as $B_1$ decreases, the magnitude of the proportionality constant increases only to 1.26 and 1.27, respectively. Finally, as suspected, when the one-dimensional assumption is valid, $B_1 \ll 1$, the effect of root variations is negligible.

Various statements may be made concerning the ratio of the heat lost by a fin when $B_1$ and $B_2$ are equal ($Q^=$) and when $B_1$ is greater than $B_2$ ($Q^{\neq}$). These relative statements are

$$\frac{Q^=}{Q^{\neq}}(B_3 = 0) > \frac{Q^=}{Q^{\neq}}(B_3 > 100) \quad \text{for all } B_1, B_2 \text{ and } b$$

$$\frac{Q^=}{Q^{\neq}}(B_2 = 0) > \frac{Q^=}{Q^{\neq}}(B_2 = B_1) \quad \text{for all } B_1, B_3 \text{ and } b$$

$$\left.\begin{aligned} &\frac{Q^=}{Q^{\neq}}(B_1 = 1) < \frac{Q^=}{Q^{\neq}}(B_1 = 0.01) \quad \text{for } B_3 = 0 \\ &\qquad\qquad \text{for all } B_2 \text{ and } b \text{ (see Table 1)} \\ &\qquad > \frac{Q^=}{Q^{\neq}}(B_1 = 0.01) \quad \text{for } B_3 > 0.25 \end{aligned}\right\}$$

$$\frac{Q^=}{Q^{\neq}}(b < 0) \sim \frac{Q^=}{Q^{\neq}}(b > 0) \quad \text{for all } B_1, B_2 \text{ and } B_3$$

and

$$\frac{Q^=}{Q^{\neq}}(B_2 < B_1) > \frac{Q^=}{Q^{\neq}}(B_2 = B_1) = 1.$$

A curious interrelationship of the surface Biot numbers exists. That is for a given $B_2/B_1$, a $B_3$ exists such that $Q^=/Q^{\neq}$ does not change with $B_1$. The range of parameters for this relationship to exist is not wide.

Finally the magnitude of the root temperature variation, $b$, through a very important parameter for $Q(\text{lost})/k\theta_0$, is not a strong parameter for the $Q^=/Q^{\neq}$ values. Thus the variation effects of the asymmetric fin, as described in this study, have nearly the same variation as those of the symmetric fin regardless of the magnitude of $b$.

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